THERMOELASTIC WAVE WITH A FINITE HEAT PROPAGATION VELOCITY

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It is shown that, when the velocity of heat propagation is considered, the thermoelasticity problem with instantaneous heating of a surface does not have a solution. A method is proposed for determining the amplitude of a thermoelastic wave from the initial conditions.

It is well known that in the conventional theory of heat conduction ones assumes the heat propagation velocity to be infinite. In dynamic thermoelasticity problems this premise leads, formally, to the appearance of stresses at a point already before the wave has reached it. Such a "blurring" of the thermal stress wave front occurs even when the temperature profile is at variance with the elastic wave (see [1]).

Introducing a finite heat propagation velocity yields a hyperbolic equation of heat conduction (see, e.g., [2]):

$$\frac{\partial T}{\partial t} + \frac{a}{v_\text{т}^2} \cdot \frac{\partial^2 T}{\partial t^2} = a \Delta T. \tag{1}$$

In dielectrics, where heat conduction through the lattice is the basic mode of heat transmission, the normal vibrations of all atoms together are treated as a phonon gas. At temperatures above the Debye point the nonharmonic interaction determines the mean free path length of the phonon and the establishment of thermal equilibrium. The velocity of phonons, i.e., of heat propagation, happens to be the velocity of sound whether through crystals or through amorphous glass [3].

In metals at moderate or high temperatures the propagation of heat is due to free electrons subject to the Fermi–Dirac statistics. It follows from this distribution that only those electrons contribute to the conduction of heat whose energy approaches the Fermi level within approximately kT [4]. The electron velocity corresponding to this energy is in most metals approximately 10^6m/sec. Such a high velocity of "Fermi" electrons is still not sufficient to ensure a high velocity of heat propagation in a metal, however, this latter velocity being that at which the temperature rise of the lattice alone is propagated. A limiting factor here is the time required for energy transfer between atoms in the metal. A rigorous analysis requires a solution of the relaxation equation for electrons and phonons. On the basis of known results [3, 5], we establish that the transit time for an electron moving at a 10^6m/sec velocity in a metal is by 2-4 orders of magnitude shorter than the electron–phonon relaxation time $t_r \sim 10^{-11}$ sec [5]. This value agrees with the time t_r which A. V. Lykov determined for aluminum [2]. The heat propagation velocity corresponding to this time

$$v_\text{т} = \sqrt{\frac{a}{t_\text{р}}}$$

amounts to several kilometers per second, i.e., is of the same order of magnitude as the velocity of sound.

We will now consider thermoelasticity problems with the heat propagation velocity taken equal to the velocity of sound, i.e., to a specific velocity in substances with phonon heat conductivity. When temperature and thermal stress perturbations travel through a substance at equal velocities, then, in essence, a wave is propagated. When the initial conditions are given in terms of discrete values (momentary thermal shock), then the process variables including pressure and temperature as well as density (within elastic strain limits) of such a wave change jumpwise. For this reason, the equations for a continuous medium, i.e., the equations of thermoelasticity, are not applicable when the state variables of that medium are to be

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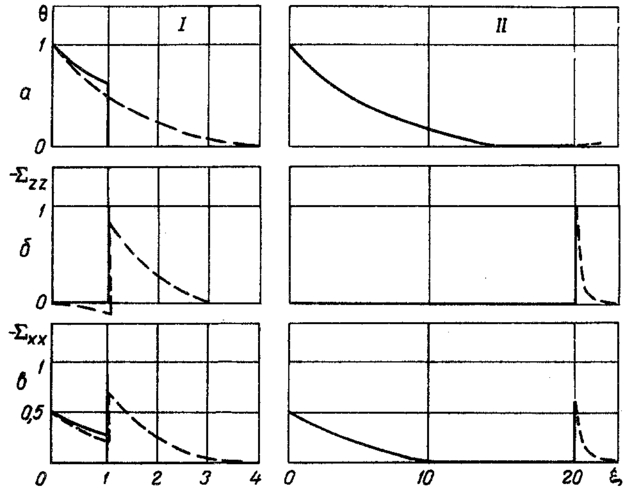

Fig. 1. Temperature and stress distributions at instants of time: $\tau=1$ (I) and $\tau=20$ (II) in dimensionless θ, Σ, ξ coordinates: (a) temperature wave θ, (b) Σ_{zz} stresses, and (c) $\Sigma_{xx}=\Sigma_{yy}$ stresses. Dashed lines refer to $v_T=\infty$ [1].

determined at a rupture point. Therefore, dynamic problems of thermoelasticity with the source "switched on" momentarily cannot, in principle, be solved without introducing additional conditions at the jump.

With an infinite heat propagation velocity, as has been assumed earlier, the stresses will change jumpwise while the temperature changes smoothly [1]. Such an "isothermal" jump is known in gas dynamics with radiative heat transfer (e.g., in [6]), provided that the thermal flux is exactly proportional to the temperature gradient, i.e., when the equation of heat conduction is parabolic. Using the hyperbolic equation (1), however, the problem as analyzed by M. D. Mikhailov in [7] involves a discontinuity precisely where $M \to 1$, i.e., there is no solution for the case where the heat propagation velocity is equal to the velocity of sound.

An approximate solution to dynamic problems of thermoelasticity with a momentary temperature rise can be obtained, in the two-dimensional as well as in the three-dimensional case, by superposing the quasi-static solution on the elastic wave specified by the initial conditions at the jump.

Let us now consider the problem of a momentary thermal shock in a half-space. When $v_T=c_0$, the hyperbolic equation (1) yields a frontal temperature jump with an amplitude which decreases according to

$$T = T_m \exp\left(\frac{-c_0^2 t}{2a}\right).$$

The form of a temperature wave with a jump is shown in Fig. 1a at two instants of time and is compared here with the solution to the parabolic temperature equation ($v_T=\infty$). The temperature jump alone remains significant before $t \lesssim 10^{-12}$ sec, its amplitude decreasing with time. We note that a temperature wave with a jump can be obtained from [7] when $M=1$ in our case.

At the instant the temperature rises, a thin surface layer of the substance is heated up without volume expansion and a thermal pressure P is generated in it momentarily. This jump of thermal pressure produces an elastic wave which travels from the surface deep into the bulk of the substance. In the two-dimensional case with dissipation disregarded the amplitude of this wave does not change throughout the process.

Since the heating produces an isotropic compression of the substance, the propagation velocity of a stress jump can be expressed in terms of the bulk modulus of compression K:

$$c_0 = \sqrt{\frac{K}{\rho}}, \text{ where } K = \frac{E}{3(1-2\nu)}.$$

In our case the strain is uniform and the per unit change of volume $\Delta V/V_0$ is related to stress σ_{zz} as follows (see, e.g., [8]):

$$\frac{\Delta V}{V_0} = -\frac{\sigma_{zz}}{3K} = -\frac{\sigma_{zz}}{E}(1-2\nu).$$

Considering that $\Delta V/V_0 = \alpha T_m$ by definition, we obtain

$$\sigma_{zz} = -\frac{E\alpha T_m}{1-2\nu}.$$

The remaining components of the stress tensor under bulk compression are σ_{zz}, as can be easily verified by inserting the value found here into the known expressions:

$$\sigma_{xx} = \sigma_{yy} = \frac{\nu}{1-\nu}\sigma_{zz} - \frac{E\alpha T_m}{1-\nu} = -\frac{E\alpha T_m}{1-2\nu}.$$

The quasistatic component of σ_{zz} is in our case equal to zero, i.e., stress σ_{zz} (Fig. 1b) is represented by a wave penetrating the substance without attenuation. The thermoelastic wave representing stresses σ_{xx} and σ_{yy} soon separates from the quasistatic component and continues to travel as a single wave (Fig. 1c).

Quasistatic stresses in the two-dimensional homogeneous case "follow" the temperature field and their maxima at the surface (z = 0) are

$$\sigma_{xx} = \sigma_{yy} = -\frac{E\alpha T_m}{1-\nu}.$$

The main feature of the solution shown in Fig. 1 for a momentary thermal shock is the steep front of the thermoelastic wave, which differs from the gradual decrease of σ_{kk} shown in earlier references and due to the formal introduction of an infinite heat propagation velocity. For comparison, we also show here temperature and stress profiles based on the parabolic equation of heat conduction. The difference of amplitudes behind the wave front does not exceed 15-30% and decreases further with time. It is physically impossible for large stresses to appear ahead of the wave [1], as the obtained solution demonstrates.

In the case of a thermal shock with a finite heating time, the front of the temperature wave does not jump and the wave does not coincide with the elastic wave. The stress jump, which travels through the substance, can then be found from the solution to the dynamic problem. Specifically, the generalization of a two-dimensional thermal shock with a finite heat propagation velocity [7] and linear heating in the time till $T = T_m$

$$t_0 = \frac{a\tau_0}{c_0^2}$$

yields

$$\sum\nolimits_{zz} = \frac{1-2\nu}{E\alpha\theta}\sigma_{zz} = \frac{1}{\tau_0(1-M^2)}\left\{\int_0^\tau \exp\left(\frac{\tau-\xi-y}{1-M^2}\right)\left[\eta(\tau-\xi-y)\right.\right.$$
$$\left. - \eta(\tau-\xi M-y)\exp\left(\frac{\xi}{2M}\,\frac{M-1}{M+1}\right)\right]dy$$
$$\left. -\frac{\xi}{2M}\int_0^\tau\int_{\xi/2M}^{(\tau-y)/2M^2} \exp\left(-x\,\frac{1+M^2}{1-M^2}\right)\frac{I_1\left[\sqrt{x^2-(\xi/2M)^2}\right]}{\sqrt{x^2-(\xi/2M)^2}}\,dxdy\right\},$$

where $M = c_0/v_T$ (c_0 is the velocity of sound), $\theta = T/T_m$, $\tau = c_0^2 t/a$, $\xi = c_0 z/a$, and T_m is the magnitude of the initial temperature jump. At the front of the elastic wave ($\tau = \xi$) we have

$$\sum\nolimits_{zz} = \frac{1}{\tau_0}\left\{\exp\left(\frac{\tau}{2M}\cdot\frac{M-1}{M+1}\right)\left[\exp\left(-\frac{\tau}{M+1}\right)-1\right]\right.$$
$$\left. -\frac{\tau}{2M(1-M^2)}\int_0^\tau\int_{\tau/2M}^{(\tau-y)/2M^2} \exp\left(-x\,\frac{1+M^2}{1-M^2}\right)\frac{I_1\left[\sqrt{x^2-(\tau/2M)^2}\right]}{\sqrt{x^2-(\tau/2M)^2}}\,dxdy,\right.$$

from which it is not difficult to show that the second term on the right-hand side becomes zero and the magnitude of the stress jump becomes

$$\sum_{zz} = -\frac{1}{\tau_0}\left[1-\exp\left(-\frac{\tau}{2}\right)\right] \tag{2}$$

when $M \to 1$. Expression (2) indicates that, when the thermal shock time τ_0 is shortened, the amplitude of stresses increases and rupture will occur when $\tau_0 = 0$.

For solids with predominant electron thermal conductivity (metals and semiconductors at high temperature), as has been shown already, the heat propagation velocity v_T is almost equal to the velocity of sound. Thus, the propagation of temperature and stress perturbations will also have the character of a weak shock (acoustic) wave whose length is determined by the kinetics of electron–phonon relaxations and of sound dispersion.

The width of the shock wave alone is determined by the difference between the velocity of sound in the original and in the compressed substance, this difference being approximately equal to $0.2c_0P/E$ in most solids. Taking into account variations in the elastic constants E and ν will add a correction of a higher order of smallness [8]. If a thermal shock excites other wave modes besides expansion waves, then their dispersion will also increase the width of the compressed layer. The profile of the thermoelastic wave alone can be determined only with the complete equation of state of the solid known.

It is to be noted that from the condition for pressure P at a jump

$$P = -\sigma_{zz} = \rho u c_0$$

we can find the velocity of the mass in the substance behind the wave:

$$u = -\frac{\sigma_{zz}}{\rho c_0} = \alpha T\sqrt{\frac{3E}{\rho(1-2\nu)}}.$$

For example, for copper ($\alpha = 1.65 \cdot 10^{-5}$ deg^{-1}, $\nu = 0.34$, and $E = 1.2 \cdot 10^5$ MN/m^2) at a temperature jump of 20°C the pressure is $P = 124$ MN/m^2 and the mass velocity is $u = 3.5$ m/sec.

NOTATION

T (θ)	is the temperature (dimensionless);
z (ξ)	is the coordinate (dimensionless);
t (τ)	is the time (dimensionless);
σ(Σ)	is the stress (dimensionless);
P	is the pressure;
a	is the thermal diffusivity;
v_T	is the propagation velocity;
c_0	is the velocity of sound;
u	is the mass velocity;
V	is the volume
k	is the Boltzmann constant;
K	is the bulk modulus of compression;
α	is the thermal expansivity;
ρ	is the density;
ν	is the Poisson ratio;
E	in Young's modulus of elasticity.

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